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ON THE KERNEL OF THE STIELTJES INTEGRAL CORRESPONDING TO A COMPLETELY CONTINUOUS TRANSFORMATION.*

BY CHARLES ALBERT FISCHER.

A large part of the Fredholm theory of integral equations has been derived for the equation

$$(1) \quad g(x) = f(x) - \lambda A(f),$$

where A is a completely continuous, linear transformation.† It is well known that every linear transformation can be put into the form

$$(2) \quad A(f) = \int_a^b f(y) d_y K(x, y),$$

and the conditions which K must satisfy in order that A shall be completely continuous have been found.‡ The present paper discusses the relation between $K(x, y)$ and the solutions of the homogeneous equation corresponding to (1). In the first section some properties of orthogonal transformations are discussed, and it is proved that the limit of a uniformly convergent sequence of completely continuous, linear transformations is of the same type. In § 2 it is proved that a null element with respect to the transformation

$$(3) \quad B_\lambda(f) = f(x) - \lambda A(f)$$

is a kernel element with respect to all other values of λ . Riesz has proved that A can be decomposed into the sum of two transformations A' and A'' , such that the transformation $B'(f) = f(x) - A'(f)$ has a unique inverse, and the equations $B^n(f) = 0$ and $B''^n(f) = 0$ have the same solutions.§ In § 3 of the present paper it is proved that the $K''(x, y)$ corresponding to A'' can be put into the form

$$(4) \quad K''(x, y) = \sum_{i=1}^n \varphi_i(x) \psi_i(y),$$

where $\varphi_1, \varphi_2, \dots, \varphi_n$ are null elements. In the next section this theory is applied to the transformation B_λ , and in the last section the Stieltjes integral equation of the first kind is discussed.

§ 1. Orthogonal Transformations and Uniformly Convergent Sequences.

* Presented to the American Mathematical Society, April 23, 1921.

† F. Riesz, *Acta Mathematica*, Vol. 41 (1918), pp. 71–98.

‡ Fischer, *Bulletin of the American Mathematical Society*, Vol. 27 (1920), pp. 10–17.

§ Riesz, loc. cit., theorems 10 and 11.

—In all that follows $A(f)$, $A'(f)$, etc., will represent completely continuous, linear transformations carrying functions of the class $\{f\}$ into other functions of the same class. The class $\{f\}$ will be composed of all bounded functions defined on the interval (a, b) , which are summable with respect to a function of bounded variation by Young's method of monotone sequences,* beginning with continuous functions. The transformations B , B' , etc., will be defined by equations such as $B = E - A$, etc., where E is the identical transformation, and B_λ by equation (3). When $A(f)$ is put into the form (2), the function $K(x, y)$ is determined uniquely when it is required to satisfy the equations $K(x, a) = 0$, and $K(x, y) = K(x, y + 0)$, $(a < y < b)$.† If $K(x, y)$, considered as a function of x , belongs to $\{f\}$ for every value of y , and $f(y)$ is continuous, the integral (2) can be approached by a bounded sequence of functions such as

$$\sum_{i=1}^n f(\eta_i) [K(x, y_i) - K(x, y_{i-1})], \quad (y_{i-1} < \eta_i < y_i),$$

each of which belongs to $\{f(x)\}$, and if f is a discontinuous function belonging to $\{f\}$ the integral (2) can be approached by a system of bounded sequences beginning with such integrals for continuous f 's. Consequently the integral (2) must also belong to $\{f\}$.‡ If $K(x, y)$ does not belong to $\{f\}$ for some value of y , the function $f(y)$ can be defined in such a way that $A(f)$ will not belong to $\{f\}$.

The necessary and sufficient condition that $A(f)$ be completely continuous is that $V_y K(x, y)$, that is the variation of K considered as a function of y , shall be bounded uniformly, and that when x_1, x_2, \dots are chosen in such a way that $K(x_r, y)$ approaches a unique limit when r becomes infinite, the equation

$$\lim_{r \rightarrow \infty} V_y [K(x_r, y) - \lim_{r \rightarrow \infty} K(x_r, y)] = 0$$

shall be satisfied.§

The transformations A_1 and A_2 are said to be orthogonal if the equations $A_1 A_2(f) = A_2 A_1(f) = 0$ are satisfied identically in the argument f .

If A_1 and A_2 are orthogonal and $B_1(f) = 0$, then $A_2(f) = A_2 A_1(f) = 0$, and if $A = A_1 + A_2$, the equation $B(f) = 0$ must also be satisfied. Under the same circumstances, it follows from definition that $B_1 + B_2 = E + B$ and that $B_1 B_2 = B$. Then if $B(f) = 0$, $B_1(f) + B_2(f) = f$, and $B_1 B_2(f) = B_2 B_1(f) = 0$. Thus putting $f_1 = B_2(f)$ and $f_2 = B_1(f)$, the following theorem has been proved; if A_1 and A_2 are orthogonal, and $A_1 + A_2 = A$,

* Young, *Proceedings of the London Mathematical Society*, Vol. 13 (1914), p. 109.

† Fischer, *Annals of Mathematics*, Vol. 19 (1917), pp. 39–40.

‡ Daniell, *Annals of Mathematics*, Vol. 19 (1918), p. 290, theorem 7 (7).

§ Fischer, *Bulletin*, loc. cit., p. 14.

the necessary and sufficient condition that $B(f) = 0$ is that it be the sum of two functions, f_1 and f_2 , such that $B_1(f_1) = B_2(f_2) = 0$. This can easily be generalized. If A_1 and A_2 are orthogonal, the equation $A^r(f) = A_1^r(f) + A_2^r(f)$ must be satisfied for $r = 1, 2, \dots$, and consequently $B_1^n + B_2^n = E + B^n$ and $B_1^n B_2^n = B^n$. Consequently, the necessary and sufficient condition that $B^n(f) = 0$ is that f be the sum of an f_1 and an f_2 such that $B_1^n(f_1) = B_2^n(f_2) = 0$.

A sequence of transformations A_1, A_2, \dots will be said to converge uniformly to a transformation A , if for every $\epsilon > 0$ there is an n_ϵ independent of f and x , such that for $n \geq n_\epsilon$ the inequality $|A_n(f) - A(f)| \leq \epsilon \|f\|$ shall be satisfied. The notation $\|f\|$ represents the least upper bound of $|f(x)|$.

It will now be proved that the limit of a uniformly convergent sequence of completely continuous, linear transformations is completely continuous and linear. The limit of such a sequence is evidently distributive, and if it can be proved to be bounded it must be linear. If the transformations are put into the form

$$A_n(f) = \int_a^b f(y) d_y K_n(x, y),$$

$V_y K_n(x, y)$ is the least upper bound of $|A_n(f)| / \|f\|$. It follows from the definition of uniform convergence that $V_y [K_n(x, y) - K(x, y)]$ approaches zero uniformly in x as n becomes infinite. If the least upper bound of $V_y K_n(x, y)$ is then called M_n , and n is taken as large as the n_ϵ mentioned above, the inequality $|A(f)| \leq (M_n + \epsilon) \|f\|$ must be satisfied. Therefore the transformation A is bounded and linear. If it were not completely continuous there would be a sequence x_1, x_2, \dots and an $\epsilon > 0$ such that $K(x_r, y)$ would approach a limiting function $k(y)$ as r became infinite, while

$$(5) \quad V_y (K(x_r, y) - k(y)) > \epsilon, \quad (r = 1, 2, \dots).$$

The function $K_1(x_r, y)$ would have to converge to a function, which will be called $k_1(y)$, for a subset of these x 's.* In the same way a subset of this subset would make $K_2(x, y)$ converge. In this way the sequences $x_1^{(n)}, x_2^{(n)}, \dots$ could be determined in such a way that each is a subset of the preceding, and $K_n(x_r^{(n)}, y)$ would converge to a $k_n(y)$ as r became infinite. Consequently the one sequence $x_1^{(1)}, x_2^{(2)}, \dots$ would make $K_n(x_r^{(r)}, y)$ converge for every n . If n were then taken large enough the inequality

$$(6) \quad V_y (K_n(x, y) - K(x, y)) < \frac{\epsilon}{4}, \quad (a \leq x \leq b),$$

would have to be satisfied, and then, since A_n is completely continuous, r could be taken so large that the inequalities

* Fischer, *Bulletin*, loc. cit., p. 13. This also has been proved by Helly.

$$(7) \quad V_y(K_n(x_r^{(r)}, y) - K_n(x_{r+i}^{(r+i)}, y)) < \frac{\epsilon}{2}, \quad (i = 1, 2, \dots),$$

would be satisfied. Inequalities (6) and (7) would imply that

$$V_y(K(x_r^{(r)}, y) - K(x_{r+i}^{(r+i)}, y)) < \epsilon, \quad (i = 1, 2, \dots),$$

and since the variation of the limit of a sequence of functions cannot be greater than the limit of their variations,* inequality (5) could not be satisfied. That is $A(f)$ must be completely continuous.

The following example illustrates the fact that the limit of a non-uniformly convergent sequence of completely continuous transformations need not be completely continuous. The functions $K_1(x, y)$, $K_2(x, y)$, \dots corresponding to A_1 , A_2 , \dots will be defined as identically zero for all values of x excepting $x_r = 1 - 1/r$, ($r = 1, 2, \dots$), and for these values of x by the equations

$$\begin{aligned} K_n(x_r, y) &= 0, & (0 \leq y < x_r), \\ K_n(x_r, y) &= 1, & (r = 1, 2, \dots, n; \quad x_r \leq y \leq 1), \\ K_n(x_r, y) &= n/r, & (r = n, n+1, \dots; \quad x_r \leq y \leq 1). \end{aligned}$$

Each of these functions satisfies the conditions for complete continuity, and as n becomes infinite the function $K(x, y)$ corresponding to the limiting transformation satisfies the equations

$$\begin{aligned} K(x_r, y) &= 0, & (0 \leq y < x_r), \\ K(x_r, y) &= 1, & (x_r \leq y \leq 1). \end{aligned}$$

Consequently

$$V_y[K(x_r, y) - \lim_{r \rightarrow \infty} K(x_r, y)] = 2,$$

and $A(f)$ cannot be completely continuous.

It will now be proved that if the corresponding terms of two uniformly convergent sequences of completely continuous transformations are orthogonal, their limits are orthogonal. The sequences will be called A_1 , A_2 , \dots and \overline{A}_1 , \overline{A}_2 , \dots . Since $A_n \overline{A}_n(f) = 0$, the inequality

$$\|A \overline{A}(f)\| \leq \|A[\overline{A}(f) - \overline{A}_n(f)]\| + \|[A - A_n]\overline{A}_n(f)\|,$$

must be satisfied. The first term of the right member of this inequality must approach zero uniformly for a bounded set of f 's, as n becomes infinite, because A is bounded and \overline{A}_n approaches \overline{A} uniformly, and the second term approaches zero for a similar reason. Therefore A and \overline{A} must be orthogonal.

§ 2. On Kernel Elements and Null Elements.—An element $f(x)$ is said to be null with respect to A if it is a solution of the equation $B^v(f) = 0$, and

* Fischer, *Bulletin*, loc. cit., p. 13.

is said to be kernel if there is a g in $\{f\}$ which satisfies the equation $B^\nu(g) = f$. The integer ν is the smallest integer such that every solution of $B^{\nu+1}(f) = 0$ is also a solution of $B^\nu(f) = 0$.*

One of the theorems in § 1 might have been stated: If A_1 and A_2 are orthogonal and $A_1 + A_2 = A$, every element which is null with respect to either A_1 or A_2 is also null with respect to A , and every element which is null with respect to A is the sum of two elements one of which is null with respect to each of the transformations A_1 and A_2 . Of course one of the two may be identically zero.

It will now be proved that if f is null with respect to $\lambda_1 A$, it must be kernel with respect to λA when $\lambda \neq \lambda_1$. To accomplish this it will first be proved that if $B_{\lambda_1}(f) = 0$, f must be kernel with respect to λ , and second that if g is kernel with respect to λ and $B_{\lambda_1}(f) = g$, f must also be kernel. Then since when $B_{\lambda_1}^2(f) = 0$, $g = B_{\lambda_1}(f)$ must be a solution of $B_{\lambda_1}(g) = 0$, every solution of $B_{\lambda_1}^2(f) = 0$, and in the same way every solution of $B_{\lambda_1}^\nu(f) = 0$, must be kernel with respect to λA . If $B(f_{\lambda_1}) = 0$, it follows from definition that $A^n(f) = f/\lambda_1^n$, and consequently

$$B_\lambda^\nu(f) = \left(1 - \frac{\lambda}{\lambda_1}\right)^\nu \cdot f.$$

That is $(1 - \lambda/\lambda_1)^\nu f$, and consequently f itself, is kernel with respect to λA . To prove the second part, it will be assumed that $B_{\lambda_1}(f) = g$, and g is kernel with respect to λA . Then by definition $\lambda_1 A[f] = f - g$, and

$$A^n[f] = \frac{1}{\lambda_1^n} \{f - g - \lambda_1 A(g) - \cdots - \lambda_1^{n-1} A^{n-1}(g)\}.$$

This makes it possible to put $B_\lambda^\nu(f)$ into the form

$$B_\lambda^\nu(f) = \left(1 - \frac{\lambda}{\lambda_1}\right)^\nu f + c_0 g + c_1 A(g) + \cdots + c_{\nu-1} A^{\nu-1}(g),$$

where c_0, c_1, \cdots are finite constants determined by λ and λ_1 . Since the left member of this equation, and every term of the right member except the first, is kernel with respect to λA , that term, and consequently f itself, must be kernel.

It has been proved† that if λ is a critical value, that is a value such that there are null elements with respect to λA , each f determines f' and f'' uniquely, such that $f = f' + f''$ and f' is kernel and f'' null with respect to λA . All of the critical values can be arranged in a sequence $\lambda_1, \lambda_2, \cdots, \ddagger$ and f can then be decomposed into

$$f = f_1' + f_2' + \cdots + f_n' + f_n'', \quad (n = 1, 2, \cdots),$$

* See Riesz, loc. cit., theorem 2.

† Riesz, loc. cit., theorem 8.

‡ Riesz, loc. cit., theorem 12.

where f'_n is kernel with respect to $\lambda_1, \lambda_2, \dots, \lambda_n$, and f''_n is null with respect to λ_n and kernel with respect to all other values of λ .

If the series $f'' = f''_1 + f''_2 + \dots$ converges uniformly, the function $f - f''$ is the limit of a uniformly convergent sequence of functions $f'_{n+1}, f'_{n+2}, \dots$, each of which is kernel with respect to λ_n . Consequently it is kernel with respect to λ_n , ($n = 1, 2, \dots$).*

§ 3, **The Kernel** $K''(x, y)$.—It is proved in Riesz' theorem 10 that the transformation A determines the orthogonal transformations A' and A'' , such that if f' is kernel and f'' null with respect to A , the equations

$$A''(f') = A'(f'') = 0, \quad A'(f') = A(f'), \quad A''(f'') = A(f''),$$

are all satisfied. His theorem 1' states that all the kernel elements can be expressed linearly in terms of a finite number of them. These can be selected in the following way. The first ones $\varphi_1, \varphi_2, \dots, \varphi_s$ will be a complete set of linearly independent solutions of the equation $B(f) = 0$. Then if $\nu > 1$ there must be one or more independent solutions of $B^2(f) = 0$, which are not solutions of $B(f) = 0$, and these will be called $\varphi_{s+1}, \varphi_{s+2}, \dots, \varphi_{s+i}$. The solutions of $B^3(f) = 0$ will follow, etc., until all the linear independent null elements are exhausted. Since when $B^n(f) = 0$, $B(f)$ is a solution of $B^{n-1}(f) = 0$, $B(\varphi_i)$ must be a linear combination of $\varphi_1, \varphi_2, \dots, \varphi_{i-1}$. The equations

$$(8) \quad B(\varphi_i) = \sum_{j=1}^{i-1} a_{ij}\varphi_j, \quad (i = s+1, s+2, \dots, r),$$

must then be satisfied, where the a_{ij} 's are constants. The φ 's will also be determined so as to satisfy the equations $\|\varphi_i\| = 1$, ($i = 1, 2, \dots, r$). Every element f can be put into the form

$$(9) \quad f(x) = f'(x) + c_1\varphi_1(x) + c_2\varphi_2(x) + \dots + c_r\varphi_r(x),$$

in one and only one way, where f' is a kernel element, and the c 's are constants. It follows from the proof of Riesz' theorem 9 that there is a constant C , independent of f and x , such that

$$\left\| \sum_{i=1}^r c_i \varphi_i \right\| \leq C \|f\|,$$

and since $\varphi_1, \varphi_2, \dots, \varphi_r$ are linearly independent, there is an M , independent of f and x , which satisfies the inequalities

$$|c_i| = \|c_i \varphi_i\| \leq M \left\| \sum_{i=1}^r c_i \varphi_i \right\|. \dagger$$

* It can be proved by means of Riesz' theorem 4 that the limit of a uniformly convergent set of kernel elements is kernel.

† See the proof of Riesz' lemma 4.

Consequently $|c_i| = MC \|f\|$. The transformation A'' can be decomposed into the sum of r transformations defined by the equations

$$(10) \quad A_i(f) = c_i A(\varphi_i), \quad (i = 1, 2, \dots, r),$$

where the c 's are given by equation (9). These transformations are distributive and satisfy the inequalities $\|A_i(f)\| \leq MCM_A \|f\|$, where M_A is the least upper bound of $|K(x, y)|$. Therefore they are linear, and since each must transform a bounded set of functions into a compact set, they are completely continuous.

The transformations A' , A'' and A_i can be expressed by the equations

$$\begin{aligned} A'(f) &= \int_a^b f(y) d_y K'(x, y), \\ A''(f) &= \int_a^b f(y) d_y K''(x, y), \\ A_i(f) &= \int_a^b f(y) d_y K_i(x, y), \quad (i = 1, 2, \dots, r), \end{aligned}$$

where the K 's are determined uniquely by the conditions given in § 1. Then it follows from definition that

$$(11) \quad K''(x, y) = \sum_{i=1}^r K_i(x, y).$$

Since $\varphi_1, \varphi_2, \dots, \varphi_s$ are solutions of $B(f) = 0$, the equations

$$(12) \quad \int_a^b \varphi_i(y) d_y K_i(x, y) = \varphi_i(x), \quad (i = 1, 2, \dots, s),$$

must be satisfied, and equations (8) are equivalent to

$$(13) \quad \int_a^b \varphi_i(y) d_y K_i(x, y) = \varphi_i(x) - \sum_{j=1}^{i-1} a_{ij} \varphi_j(x), \quad (i = s+1, \dots, r).$$

For convenience φ_i^* will be defined as φ_i if $i \leq s$, and as $\varphi_i - \sum_{j=1}^{s-1} a_{ij} \varphi_j$ if $i > s$. This reduces equations (12) and (13) to the one form

$$(14) \quad \int_a^b \varphi_i(y) d_y K_i(x, y) = \varphi_i^*(x), \quad (i = 1, 2, \dots, r).$$

The equations

$$(15) \quad \int_a^b \varphi_i(y) d_y K_j(x, y) = 0, \quad (i \neq j),$$

also follow from the definition of A_i , and in general

$$(16) \quad \int_a^b f(y) d_y K_i(x, y) = c_i \varphi_i^*(x).$$

Consequently this integral must vanish identically in f for all values of x for which $\varphi_i^*(x) = 0$, and therefore $K_i(x, y)$ must vanish identically in y for all such values of x . The function $\psi_i(x, y)$ is then determined uniquely by the equation $K_i(x, y) = \varphi_i^*(x)\psi_i(x, y)$, excepting for values of x for which both members vanish, and for such values it can be defined arbitrarily. Equation (16) then becomes equivalent to

$$\int_a^b f(y) d_y \psi_i(x, y) = c_i.$$

It follows that if $\varphi_i^*(x_1) \neq 0$, the equation $\psi_i(x, y) = \psi_i(x_1, y)$ must be satisfied for all values of x for which $\varphi_i^*(x) \neq 0$, and $\psi_i(x, y)$ can be defined so that it will also be satisfied if $\varphi_i^*(x) = 0$. Thus $\psi_i(x, y)$ is independent of x , and that argument will be dropped. This reduces equation (11) to the form

$$(17) \quad K''(x, y) = \sum_{i=1}^r \varphi_i^*(x) \psi_i(y),$$

and equations (14) and (15) become equivalent to

$$(18) \quad \int_a^b \varphi_i(x) d\psi_j(y) = \delta_{ij}, \quad (i, j = 1, 2, \dots, r),$$

where $\delta_{ij} = 1$ or 0 , according as $i = j$ or $i \neq j$.

§ 4. Application to the Transformation B_λ .—The theory developed in the preceding section can be applied immediately to the transformation $B_\lambda = E - \lambda A$. If the critical values are represented by $\lambda_1, \lambda_2, \dots$, equation (17) can be replaced by

$$K''_\alpha(x, y) = \frac{1}{\lambda_\alpha} \sum_{i=1}^{r_\alpha} \varphi_{\alpha i}^*(x) \psi_{\alpha i}(y), \quad (\alpha = 1, 2, \dots),$$

where the $\varphi_{\alpha i}^*$ are null elements with respect to $\lambda_\alpha A$ determined as in § 3, and equation (18) by

$$(19) \quad \int_a^b \varphi_{\alpha i}(y) d\psi_{\alpha j}(y) = \delta_{ij}, \quad (i, j = 1, 2, \dots, r_\alpha).$$

The transformations $A_{\alpha i}$, corresponding to the A_i in the preceding section, vanish for elements which are kernel with respect to λ_α , and reduce null elements to null elements. It has been proved in § 2 that an element which is null for one value of λ is kernel for every other value. Consequently the equations

$$(20) \quad \int_a^b \varphi_{\alpha i}(y) d\psi_{\beta j}(y) = 0, \quad (i = 1, 2, \dots, r_\alpha; j = 1, 2, \dots, r_\beta; \beta \neq \alpha),$$

must be satisfied. Similarly each of the transformations

$$A''_{\alpha}(f) = \int_a^b f(y) d_y K''_{\alpha}(x, y), \quad (\alpha = 1, 2, \dots),$$

is orthogonal to each of the others. Since by definition A''_{α} is orthogonal to $A - A''_{\alpha}$, it must be orthogonal to $A - \sum_{\beta=1}^n A''_{\beta}$, ($n = \alpha, \alpha + 1, \dots$). It now follows from a theorem in § 1 that the transformation $E - \lambda[A - \sum_{\alpha=1}^n A''_{\alpha}]$ is regular for all values of λ excepting $\lambda_{n+1}, \lambda_{n+2}, \dots$, where it has the same null elements as B_{λ} .

If the series

$$\sum_{\alpha=1}^{\infty} K''_{\alpha}(x, y) = \sum_{\alpha=1}^{\infty} \frac{1}{\lambda_{\alpha}} \sum_{i=1}^{\tau_{\alpha}} \varphi_{\alpha i}^*(x) \psi_{\alpha i}(y),$$

converges in such a way that

$$\lim_{n \rightarrow \infty} V_y \left[\sum_{\alpha=n}^{\infty} K''_{\alpha}(x, y) \right] = 0$$

uniformly, the series of transformations $\sum_{\alpha=1}^{\infty} A''_{\alpha}$ will converge uniformly, and vice versa. Then since $\sum_{\alpha=1}^n A''_{\alpha}$ is orthogonal to $A - \sum_{\alpha=1}^n A''_{\alpha}$ for all values of n , their limits must also be orthogonal, and also $A - \sum_{\alpha=1}^{\infty} A''_{\alpha}$ must be orthogonal to each of the transformations A''_{α} . Consequently all of the null elements with respect to B_{λ} are null with respect to $E - \lambda \sum_{\alpha=1}^{\infty} A''_{\alpha}$, and the transformation $E - \lambda[A - \sum_{\alpha=1}^{\infty} A''_{\alpha}]$ is regular for all values of λ .

§ 5. The Stieltjes Integral Equation of the First Kind.—In this section it will be assumed that $g(x)$ is a known function which can be expanded into a uniformly convergent series

$$(21) \quad g(x) = \sum_{\alpha=1}^{\infty} \sum_{i=1}^{\tau_{\alpha}} g_{\alpha i} \varphi_{\alpha i}(x),$$

where the $g_{\alpha i}$'s are constants, and conditions will be found under which there is a solution of the equation

$$(22) \quad g(x) = \int_a^b f(y) d_y K(x, y),$$

which can be expanded into a similar convergent series

$$(23) \quad f(x) = \sum_{\alpha=1}^{\infty} \sum_{i=1}^{\tau_{\alpha}} f_{\alpha i} \varphi_{\alpha i}(x).$$

The function $K(x, y)$ will be assumed to satisfy the conditions for complete continuity. The constants $g_{\alpha i}$ can be determined from the equations

$$\int_a^b g(x) d\psi_{\alpha i}(x) = \int_a^b \sum_{\beta=1}^{\infty} \sum_{j=1}^{\tau_{\beta}} g_{\beta j} \varphi_{\beta j}(x) d\psi_{\alpha i}(x).$$

The series in the right members of the above are uniformly convergent, and consequently can be integrated term by term, and equations (19) and (20) reduce them to

$$\int_a^b g(x) d\psi_{\alpha i}(x) = g_{\alpha i}, \quad (i = 1, 2, \dots, r_{\alpha}; \alpha = 1, 2, \dots).$$

If equations (21) and (23) are substituted in equation (22), and the right member integrated term by term, it becomes

$$\sum_{\alpha=1}^{\infty} \sum_{i=1}^{r_{\alpha}} g_{\alpha i} \varphi_{\alpha i}(x) = \sum_{\alpha=1}^{\infty} \sum_{i=1}^{r_{\alpha}} f_{\alpha i} A(\varphi_{\alpha i}) = \sum_{\alpha=1}^{\infty} \frac{1}{\lambda_{\alpha}} \sum_{i=1}^{r_{\alpha}} f_{\alpha i} \varphi_{\alpha i}^*(x),$$

with the $\varphi_{\alpha i}^*$ defined as in § 3 by equations such as

$$\varphi_{\alpha i}^* = \varphi_{\alpha i} - \sum_{j=1}^{i-1} a_{\alpha ij} \varphi_{\alpha j}.$$

The constants $f_{\alpha i}$ are then determined by the systems of linear equations

$$f_{\alpha j} - \sum_{i=j+1}^{r_{\alpha}} a_{\alpha ij} f_{\alpha i} = \lambda_{\alpha} g_{\alpha j}, \quad (j = 1, 2, \dots, r_{\alpha}; \alpha = 1, 2, \dots).$$

Since the determinant of each system is unity, the constants $f_{\alpha i}$ are determined uniquely, and if the series (23) converges uniformly, it will furnish the required solution of equation (22).

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